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EDEXCEL INTERNATIONAL GCSE (9–1)

FURTHER PURE MATHEMATICS

Student Book

eBook included

Ali Datoo

Pearson

PEARSON EDEXCEL INTERNATIONAL GCSE (9–1)

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Published by Pearson Education Limited, 80 Strand, London, WC2R 0RL.

www.pearsonglobalschools.com

Copies of official specifications for all Pearson qualifications may be found on the website: https://qualifications.pearson.com

Edited by Linnet Bruce, Keith Gallick and Susan Lyons
Designed by Cobalt id
Typeset by Tech-Set Ltd, Gateshead, UK
Cover design by Pearson Education Limited
Picture research by Debbie Gallagher
Cover photo © Getty Images: Naahushi Kavuri / EyeEm

Inside front cover: **Shutterstock.com:** Dmitry Lobanov

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First published 2017

20 19 18 17

10 9 8 7 6 5 4 3 2

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

ISBN 978 0 435 18854 2

Printed in Slovakia by Neografia

Acknowledgements
The author and publisher would like to thank the following individuals and organisations for permission to reproduce photographs:

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CHAPTER 1

CHAPTER 2

CHAPTER 3

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CHAPTER 6

ABOUT THIS BOOK

This book is written for students following the Pearson Edexcel International GCSE (9–1) Further Pure Maths specification and covers both years of the course. The specification and sample assessment materials for Further Pure Maths can be found on the Pearson Qualifications website.

In each chapter, there are concise explanations and worked examples, plus numerous exercises that will help you build up confidence.

There are also exam practice questions and a chapter summary to help with exam preparation. Answers to all exercises are included at the back of the book as well as a glossary of Maths-specific terminology.

Points of Interest put the maths you are about to learn in a real-world context.

Learning Objectives show what you will learn in each chapter.

Hint boxes give you tips and reminders.

2 THE QUADRATIC FUNCTION

The path followed by a bottlenose dolphin jumping out of the water is called a parabola. A parabola is a **visual realisation** of the quadratic function $y = -x^2 + k$. Using this formula, scientists can calculate the height of a dolphin's jump (on the y-axis) and the distance travelled (on the x-axis).

There is no scientific agreement about why dolphins jump. Some scientists believe it is because they are trying to conserve energy, some believe it is to help them find food, and others believe they do it just for fun.

The parabola is a beautiful and elegant shape, commonly seen in nature. It is also seen in many man-made structures such as bridges and buildings.

LEARNING OBJECTIVES

- **Factorise** quadratic expressions where the **coefficient** of x^2 is greater than 1
- Complete the square and use this to solve quadratic equations
- Solve quadratic equations using the quadratic formula
- Understand and use the discriminant to identify whether the roots are (i) equal and real, (ii) unequal and real or (iii) not real
- Understand the roots α and β and know how to use them

STARTER ACTIVITIES

1 ▶ Factorise

a $6x^2 + 9x$ **b** $2b^2 + 8b$ **c** $9qm^2 - 27m$

d $9xy^2 + 36x^2y$ **e** $24x - 64x^2$

2 ▶ Factorise

a $x^2 + 9x + 18$ **b** $x^2 - 7x + 12$ **c** $x^2 - 2x - 3$

d $x^2 + 15x + 36$ **e** $x^2 + 12x + 27$

3 ▶ Factorise

a $x^2 - 9$ **b** $x^2 - 25$

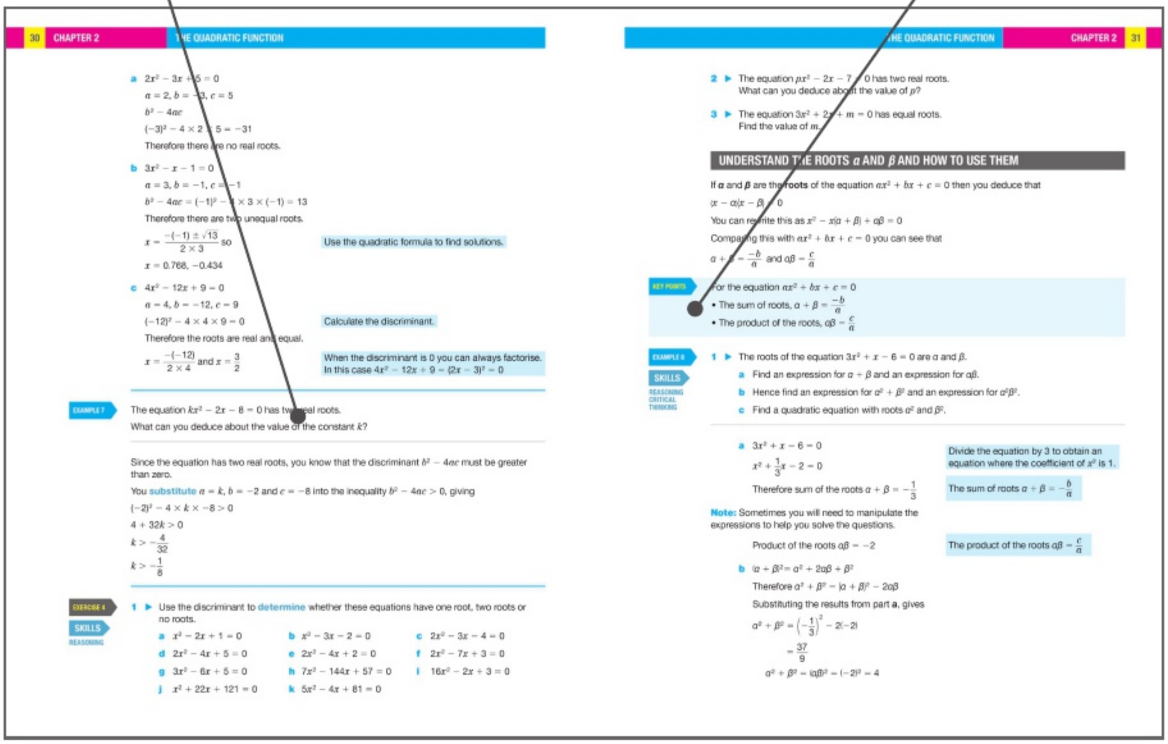
c $9x^2 - 16$ **d** $25x^2 - 16$

HINT
All the parts in question 3 are examples of the difference of two squares.

Examples provide a clear, instructional framework. The blue highlighted text gives further explanation of the method.

Language is graded for speakers of English as an additional language (EAL), with advanced Maths-specific terminology highlighted and defined in the glossary at the back of the book.

Key Points boxes summarise the essentials.



30 CHAPTER 2 THE QUADRATIC FUNCTION

a $2x^2 - 3x + 5 = 0$

$a = 2, b = -3, c = 5$

$b^2 - 4ac$

$(-3)^2 - 4 \times 2 \times 5 = -31$

Therefore there are no real roots.

b $3x^2 - x - 1 = 0$

$a = 3, b = -1, c = -1$

$b^2 - 4ac = (-1)^2 - 4 \times 3 \times (-1) = 13$

Therefore there are two unequal roots.

$x = \frac{-(-1) \pm \sqrt{13}}{2 \times 3}$ so — Use the quadratic formula to find solutions.

$x = 0.768, -0.434$

c $4x^2 - 12x + 9 = 0$

$a = 4, b = -12, c = 9$

$(-12)^2 - 4 \times 4 \times 9 = 0$ — Calculate the discriminant.

Therefore the roots are real and equal.

$x = \frac{-(-12)}{2 \times 4}$ and $x = \frac{3}{2}$ — When the discriminant is 0 you can always factorise. In this case $4x^2 - 12x + 9 = (2x - 3)^2 = 0$

EXAMPLE 7 The equation $kx^2 - 2x - 8 = 0$ has two real roots. What can you deduce about the value of the constant k?

Since the equation has two real roots, you know that the discriminant $b^2 - 4ac$ must be greater than zero.

You substitute $a = k, b = -2$ and $c = -8$ into the inequality $b^2 - 4ac > 0$, giving

$(-2)^2 - 4 \times k \times -8 > 0$

$4 + 32k > 0$

$k > -\frac{4}{32}$

$k > -\frac{1}{8}$

EXERCISE 4 SKILLS REASONING

1 ▶ Use the discriminant to determine whether these equations have one root, two roots or no roots.

a $x^2 - 2x + 1 = 0$ b $x^2 - 3x - 2 = 0$ c $2x^2 - 3x - 4 = 0$

d $2x^2 - 4x + 5 = 0$ e $2x^2 - 4x + 2 = 0$ f $2x^2 - 7x + 3 = 0$

g $3x^2 - 6x + 5 = 0$ h $7x^2 - 144x + 57 = 0$ i $16x^2 - 2x + 3 = 0$

j $x^2 + 22x + 121 = 0$ k $5x^2 - 4x + 81 = 0$

THE QUADRATIC FUNCTION CHAPTER 2 31

2 ▶ The equation $px^2 - 2x - 7 = 0$ has two real roots. What can you deduce about the value of p?

3 ▶ The equation $3x^2 + 2x + m = 0$ has equal roots. Find the value of m.

UNDERSTAND THE ROOTS α AND β AND HOW TO USE THEM

If α and β are the roots of the equation $ax^2 + bx + c = 0$ then you deduce that $(x - \alpha)(x - \beta) = 0$

You can rewrite this as $x^2 - x(\alpha + \beta) + \alpha\beta = 0$

Comparing this with $ax^2 + bx + c = 0$ you can see that

$\alpha + \beta = \frac{-b}{a}$ and $\alpha\beta = \frac{c}{a}$

KEY POINTS For the equation $ax^2 + bx + c = 0$

- The sum of roots, $\alpha + \beta = \frac{-b}{a}$
- The product of the roots, $\alpha\beta = \frac{c}{a}$

EXAMPLE 8 SKILLS REASONING CRITICAL THINKING

1 ▶ The roots of the equation $3x^2 + x - 6 = 0$ are α and β.

a Find an expression for $\alpha + \beta$ and an expression for $\alpha\beta$.

b Hence find an expression for $\alpha^2 + \beta^2$ and an expression for $\alpha^2\beta^2$.

c Find a quadratic equation with roots α^2 and β^2.

a $3x^2 + x - 6 = 0$

$x^2 + \frac{1}{3}x - 2 = 0$ — Divide the equation by 3 to obtain an equation where the coefficient of x^2 is 1.

Therefore sum of the roots $\alpha + \beta = -\frac{1}{3}$ — The sum of roots $\alpha + \beta = -\frac{b}{a}$

Note: Sometimes you will need to manipulate the expressions to help you solve the questions.

Product of the roots $\alpha\beta = -2$ — The product of the roots $\alpha\beta = \frac{c}{a}$

b $(\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$

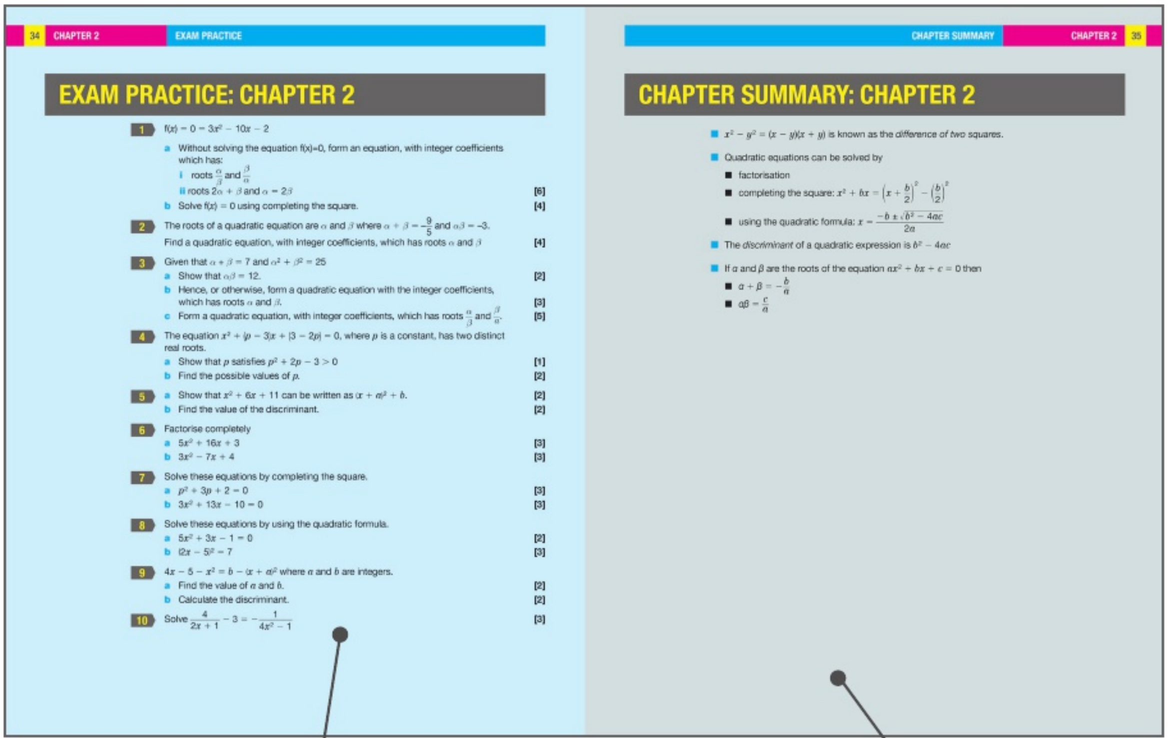
Therefore $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

Substituting the results from part a, gives

$\alpha^2 + \beta^2 = \left(-\frac{1}{3}\right)^2 - 2(-2)$

$= \frac{37}{9}$

$\alpha^2 + \beta^2 = (\alpha\beta)^2 = (-2)^2 = 4$



34 CHAPTER 2 EXAM PRACTICE

EXAM PRACTICE: CHAPTER 2

1 $f(x) = 0 - 3x^2 - 10x - 2$

a Without solving the equation f(x)=0, form an equation, with integer coefficients which has:

i roots $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$

ii roots $2\alpha + \beta$ and $\alpha + 2\beta$ [6]

b Solve f(x) = 0 using completing the square. [4]

2 The roots of a quadratic equation are α and β where $\alpha + \beta = -\frac{9}{5}$ and $\alpha\beta = -3$. Find a quadratic equation, with integer coefficients, which has roots α and β [4]

3 Given that $\alpha + \beta = 7$ and $\alpha^2 + \beta^2 = 25$

a Show that $\alpha\beta = 12$. [2]

b Hence, or otherwise, form a quadratic equation with the integer coefficients, which has roots α and β. [3]

c Form a quadratic equation, with integer coefficients, which has roots $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$. [5]

4 The equation $x^2 + (p - 3)x + (3 - 2p) = 0$, where p is a constant, has two distinct real roots.

a Show that p satisfies $p^2 + 2p - 3 > 0$ [1]

b Find the possible values of p. [2]

5 a Show that $x^2 + 6x + 11$ can be written as $(x + a)^2 + b$. [2]

b Find the value of the discriminant. [2]

6 Factorise completely

a $5x^2 + 16x + 3$ [3]

b $3x^2 - 7x + 4$ [3]

7 Solve these equations by completing the square.

a $p^2 + 3p + 2 = 0$ [3]

b $3x^2 + 13x - 10 = 0$ [3]

8 Solve these equations by using the quadratic formula.

a $5x^2 + 3x - 1 = 0$ [2]

b $(2x - 5)^2 = 7$ [3]

9 $4x - 5 - x^2 = b - (x + a)^2$ where a and b are integers.

a Find the value of a and b. [2]

b Calculate the discriminant. [2]

10 Solve $\frac{4}{2x + 1} - 3 = -\frac{1}{4x^2 - 1}$ [3]

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CHAPTER SUMMARY: CHAPTER 2

- $x^2 - y^2 = (x - y)(x + y)$ is known as the *difference of two squares*.
- Quadratic equations can be solved by
  - factorisation
  - completing the square: $x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$
  - using the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- The *discriminant* of a quadratic expression is $b^2 - 4ac$
- If α and β are the roots of the equation $ax^2 + bx + c = 0$ then
  - $\alpha + \beta = -\frac{b}{a}$
  - $\alpha\beta = \frac{c}{a}$

Exam Practice tests cover the whole chapter and provide quick, effective feedback on your progress.

Chapter Summaries state the most important points of each chapter.

ASSESSMENT OVERVIEW

The following tables give an overview of the assessment for the Edexcel International GCSE in Further Pure Mathematics.

We recommend that you study this information closely to help ensure that you are fully prepared for this course and know exactly what to expect in the assessment.

PAPER 1	PERCENTAGE	MARK	TIME	AVAILABILITY
Written examination paper Paper code 4PM1/01C Externally set and assessed by Edexcel	50%	100	2 hours	January and June examination series First assessment June 2019
PAPER 2	**PERCENTAGE**	**MARK**	**TIME**	**AVAILABILITY**
Written examination paper Paper code 4PM1/02 Externally set and assessed by Edexcel	50%	100	2 hours	January and June examination series First assessment June 2019

CONTENT SUMMARY

- Number
- Algebra and calculus
- Geometry and calculus

ASSESSMENT

- Each paper will consist of around 11 questions with varying mark allocations per questions, which will be stated on the paper
- Each paper will contain questions from any part of the specification content, and the solution of any questions may require knowledge of more than one section of the specification content
- A formulae sheet will be included in the written examinations
- A calculator may be used in the examinations

ASSESSMENT OBJECTIVES AND WEIGHTINGS

ASSESSMENT OBJECTIVE	DESCRIPTION	% IN INTERNATIONAL GCSE
AO1	Demonstrate a confident knowledge of the techniques of pure mathematics required in the specification	30%–40%
AO2	Apply a knowledge of mathematics to the solutions of problems for which an immediate method of solution is not available and which may involve knowledge of more than one topic in the specification	20%–30%
AO3	Write clear and accurate mathematical solutions	35%–50%

RELATIONSHIP OF ASSESSMENT OBJECTIVES TO UNITS

UNIT NUMBER	ASSESSMENT OBJECTIVE		
	AO1	AO2	AO3
Paper 1	15%–20%	10%–15%	17.5%–25%
Paper 2	15%–20%	10%–15%	17.5%–25%
Total for International GCSE	30%–40%	20%–30%	35%–50%

ASSESSMENT SUMMARY

The Edexcel International GCSE in Further Pure Mathematics requires students to demonstrate application and understanding of the following topics.

Number
- Use numerical skills in a purely mathematical way and in real-life situations.

Algebra and calculus
- Use algebra and calculus to set up and solve problems.
- Develop competence and confidence when manipulating mathematical expressions.
- Construct and use graphs in a range of situations.

Geometry and trigonometry
- Understand the properties of shapes, angles and transformations.
- Use vectors and rates of change to model situations.
- Use coordinate geometry.
- Use trigonometry.

Students will be expected to have a thorough knowledge of the content common to the Pearson Edexcel International GCSE in Mathematics (Specification A) (Higher Tier) or Pearson Edexcel International GCSE in Mathematics (Specification B).

Questions may be set which assumes knowledge of some topics covered in these specifications, however knowledge of statistics and matrices will not be required.
Students will be expected to carry out arithmetic and algebraic manipulation, such as being able to change the subject of a formula and evaluate numerically the value of any variable in a formula, given the values of the other variables.
The use and notation of set theory will be adopted where appropriate.

CALCULATORS

Students will be expected to have access to a suitable electronic calculator for all examination papers. The electronic calculator should have these functions as a minimum:
$+, -, \times, \div, \pi, x^2, \sqrt{x}, \frac{1}{x}, x^y$, $\ln x$, e^x, sine, cosine and tangent and their inverses in degrees and decimals of a degree or radians.

Prohibitions
Calculators with any of the following facilities are prohibited in all examinations:
databanks
retrieval of text or formulae
QWERTY keyboards
built-in symbolic algebra manipulations
symbolic differentiation or integration.

FORMULAE SHEET

These formulae will be provided for you during the examination.

MENSURATION

Surface area of sphere $= 4\pi r^2$

Curved surface area of cone $= \pi r \times$ slant height

Volume of sphere $= \frac{4}{3}\pi r^3$

SERIES

Arithmetic series

Sum to n terms $S_n = \frac{n}{2}[2a + (n - 1)d]$

Geometric series

Sum to n terms, $S_n = \frac{a(1 - r^n)}{(1 - r)}$

Sum to infinity, $S_\infty = \frac{a}{1 - r}$ $\quad |r| < 1$

Binomial series

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \quad \text{for } |x| < 1, n \in \mathbb{Q}$$

CALCULUS

Quotient rule (differentiation)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{f}(x)}{\mathrm{g}(x)}\right) = \frac{\mathrm{f}'(x)\mathrm{g}(x) - \mathrm{f}(x)\mathrm{g}'(x)}{[\mathrm{g}(x)]^2}$$

TRIGONOMETRY

Cosine rule

In triangle ABC: $a^2 = b^2 + c^2 - 2bc\cos A$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\sin(A - B) = \sin A\cos B - \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\cos(A - B) = \cos A\cos B + \sin A\sin B$

$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$

LOGARITHMS

$$\log_a x = \frac{\log_b x}{\log_b a}$$

FORMULAE TO KNOW

The following are formulae that you are expected to know and remember during the examination. These formulae **will not** be provided for you. Note that this list is not exhaustive.

LOGARITHMIC FUNCTIONS AND INDICES

$\log_a xy = \log_a x + \log_a y$

$\log_a \frac{x}{y} = \log_a x - \log_a y$

$\log_a x^k = k \log_a x$

$\log_a \frac{1}{x} = -\log_a x$

$\log_a a = 1$

$\log_a 1 = 0$

$\log_a b = \frac{1}{\log_b a}$

QUADRATIC EQUATIONS

$ax^2 + bx + c = 0$ has roots given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

When the roots of $ax^2 + bx + c = 0$ are α and β then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$ and the equation can be written $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

SERIES

Arithmetic series: nth term $= l = a + (n - 1)d$

Geometric series: nth term $= ar^{n-1}$

COORDINATE GEOMETRY

The gradient of the line joining two points (x_1, y_1) and (x_2, y_2) is $\frac{y_2 - y_1}{x_2 - x_1}$

The distance d between two points (x_1, y_1) and (x_2, y_2) is given by
$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

The coordinates of the point dividing the line joining (x_1, y_1) and (x_2, y_2) in the ratio $m : n$ are
$\left(\frac{nx_1 - mx_2}{m + n}, \frac{ny_1 - my_2}{m + n}\right)$

CALCULUS

Differentiation:

function	derivative
x^n	nx^{n-1}
$\sin ax$	$a\cos ax$
$\cos ax$	$-a\sin ax$
e^{ax}	ae^{ax}
$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$
$f(g(x))$	$f'(g(x))g'x$

Integration:

function	derivative
x^n	$\frac{1}{n+1}x^{n+1} + c \quad n \neq -1$
$\sin ax$	$-\frac{1}{a}\cos ax + c$
$\cos ax$	$\frac{1}{a}\sin ax + c$
e^{ax}	$\frac{1}{a}e^{ax} + c$

AREA AND VOLUME

Area between a curve and the x-axis $= \int_a^b y\,dx,\ y \geqslant 0$

$\left|\int_a^b y\,dx\right|,\ y < 0$

Area between a curve and the y-axis $= \int_c^d x\,dy,\ x \geqslant 0$

$\left|\int_c^d x\,dy\right|,\ x < 0$

Area between $g(x)$ and $f(x) = \int_a^b |g(x) - f(x)|\,dx$

Volume of revolution $= \int_a^b \pi y^2\,dx$ or $\int_c^d \pi x^2\,dy$

TRIGONOMETRY

Radian measure: length of arc $= r\theta$

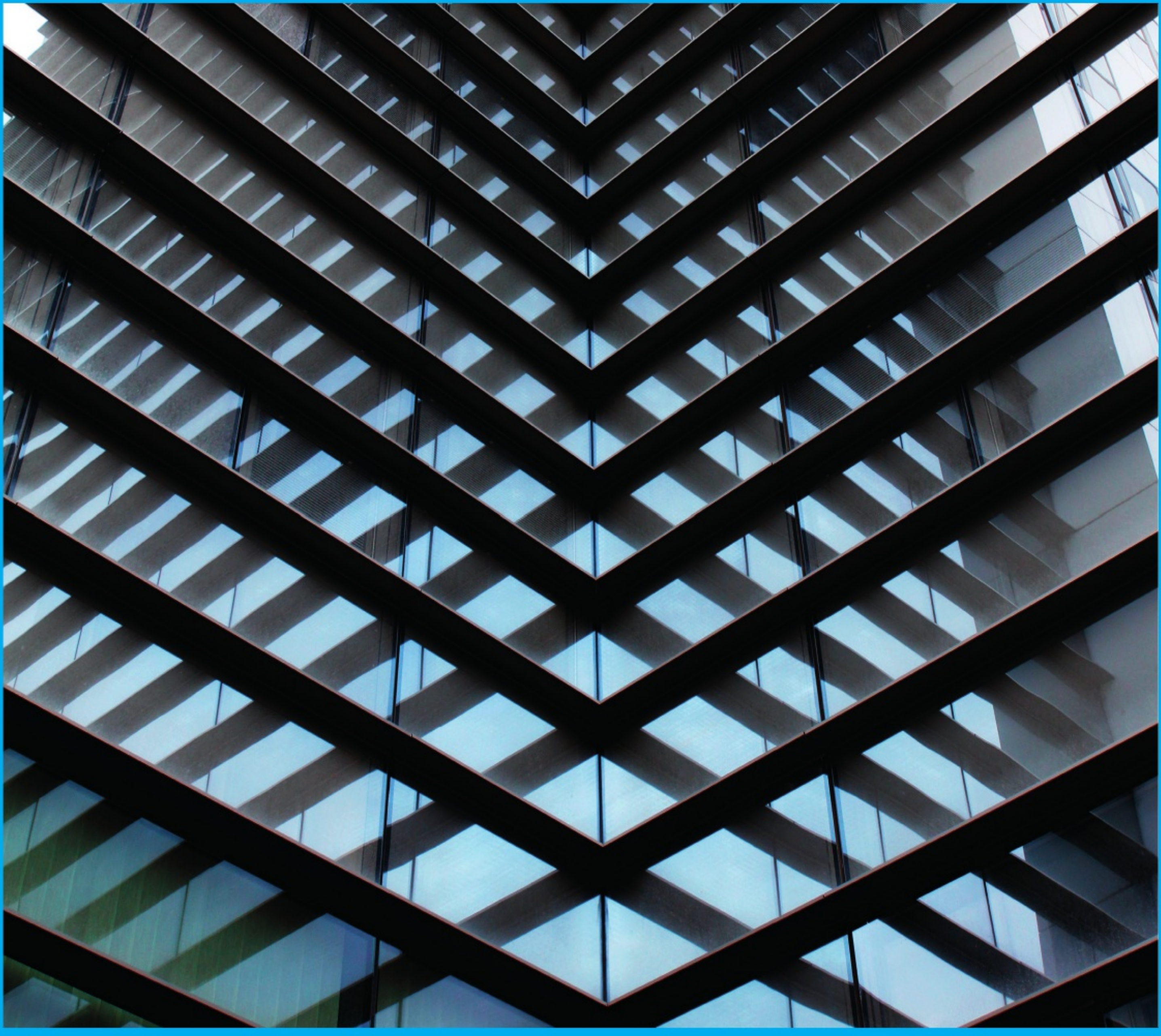
area of sector $= \frac{1}{2}r^2\theta$

In a triangle ABC: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$\cos^2\theta + \sin^2\theta = 1$

area of a triangle $= \frac{1}{2}ab\sin C$

CHAPTER 1



1 SURDS AND LOGARITHMIC FUNCTIONS

The Richter scale, which describes the energy released by an earthquake, uses the base 10 logarithm as its unit. An earthquake of **magnitude** 9 is 10 times as powerful as one of magnitude 8, and 100 000 times as powerful as one of magnitude 4.

The **devastating** 2004 earthquake in the Indian Ocean had a magnitude of 9. Thankfully, such events are rare. The most common earthquakes, which occur over 100 000 times a year, are magnitude 2 to 3, so humans can hardly feel them.

LEARNING OBJECTIVES

- Write a number exactly using surds
- Rationalise the denominator of a surd
- Be familiar with the **functions** a^x and $\log_b x$ and recognise the shapes of their graphs
- Be familiar with functions including e^x and similar terms, and use them in graphs
- Use graphs of functions to solve equations
- Rewrite expressions including powers using logarithms instead
- Understand and use the laws of logarithms
- Change the base of a logarithm
- Solve equations of the form $a^x = b$

STARTER ACTIVITIES

1 ▶ Simplify

a $y^6 \times y^5$ **b** $2q^3 \times 4q^4$ **c** $3k^2 \times 3k^7 \times 3k^{-3}$

d $(x^2)^4$ **e** $(a^4)^2 \div a^3$ **f** $64x^4y^6 \div 4xy^2$

2 ▶ Simplify

a $(m^3)^{\frac{1}{2}}$ **b** $3p^{\frac{1}{2}} \times p^3$ **c** $28c^{\frac{2}{3}} \div 7c^{\frac{1}{3}}$

d $6b^{\frac{1}{2}} \times 3b^{-\frac{1}{2}}$ **e** $27p^{\frac{2}{3}} \div 9p^{\frac{1}{6}}$ **f** $5y^6 \times 3y^{-7}$

3 ▶ Evaluate

a $16^{\frac{1}{2}}$ **b** $125^{\frac{1}{3}}$ **c** 8^{-2} **d** $(-2)^{-3}$

e $\left(\frac{6}{7}\right)^0$ **f** 81^{-4} **g** $\left(\frac{9}{16}\right)^{-\frac{3}{2}}$

WRITE A NUMBER EXACTLY USING A SURD

A surd is a number that cannot be simplified to remove a square root (or a cube root, fourth root etc). Surds are irrational numbers.

HINT
An irrational number is a number that cannot be expressed as a fraction, for example π.

NUMBER	DECIMAL	IS IT A SURD?
$\sqrt{1}$	1	No
$\sqrt{2}$	1.414213...	Yes
$\sqrt{4}$	2	No
$\sqrt{\frac{1}{4}}$	0.5	No
$\sqrt{\frac{2}{3}}$	0.816496...	Yes

You can **manipulate** surds using these rules:

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b}$$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

EXAMPLE 1

SKILLS
CRITICAL THINKING

Simplify

a $\sqrt{12}$ **b** $\frac{\sqrt{20}}{2}$ **c** $5\sqrt{6} - 2\sqrt{24} + \sqrt{294}$

a $\sqrt{12}$

$= \sqrt{4 \times 3}$

$= \sqrt{4} \times \sqrt{3}$ Use the rule $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$

$= 2\sqrt{3}$ $\sqrt{4} = 2$

b $\frac{\sqrt{20}}{2}$ $\sqrt{20} = \sqrt{4} \times \sqrt{5}$

$= \frac{\sqrt{4 \times 5}}{2}$ $\sqrt{4} = 2$

$= \frac{2 \times \sqrt{5}}{2}$

$= \sqrt{5}$

c $5\sqrt{6} - 2\sqrt{24} + \sqrt{294}$

$= 5\sqrt{6} - 2\sqrt{6}\sqrt{4} + \sqrt{6}\sqrt{49}$ $\sqrt{6}$ is a common factor

$= \sqrt{6}(5 - 2\sqrt{4} + \sqrt{49})$ Work out the square roots $\sqrt{4}$ and $\sqrt{49}$

$= \sqrt{6}(5 - 2 \times 2 + 7)$ $5 - 4 + 7 = 8$

$= 8\sqrt{6}$

EXERCISE 1

SKILLS

CRITICAL THINKING

1 ▶ Simplify without using a calculator

a $\sqrt{18}$ b $\sqrt{50}$ c $\sqrt{125}$

d $\sqrt{128}$ e $\sqrt{132}$ f $\sqrt{8625}$

2 ▶ Simplify without using a calculator

a $\frac{\sqrt{60}}{2}$ b $\frac{\sqrt{135}}{2}$ c $\frac{\sqrt{128}}{8}$

d $\frac{\sqrt{68}}{4}$ e $\frac{\sqrt{96}}{6}$

3 ▶ Simplify without using a calculator

a $6\sqrt{3} - 2\sqrt{3}$ b $7\sqrt{3} - \sqrt{12} + \sqrt{48}$ c $\sqrt{112} + 2\sqrt{172} - \sqrt{63}$

d $6\sqrt{48} - 3\sqrt{12} + 2\sqrt{27}$ e $3\sqrt{578} - \sqrt{162} + 4\sqrt{32}$ f $2\sqrt{5} \times 3\sqrt{5}$

g $6\sqrt{7} \times 4\sqrt{7}$ h $4\sqrt{8} \times 6\sqrt{8}$

4 ▶ Simplify without using a calculator

a $6(4 - \sqrt{12})$ b $9(6 - 3\sqrt{29})$ c $4(1 + \sqrt{3}) + 3(3 + 2\sqrt{3})$

d $3(\sqrt{2} - \sqrt{7}) - 5(\sqrt{2} + \sqrt{7})$ e $(4 + \sqrt{3})(4 - \sqrt{3})$

f $(2\sqrt{7} - \sqrt{6})(\sqrt{7} - 2\sqrt{6})$ g $(\sqrt{8} + \sqrt{5})(\sqrt{8} - \sqrt{5})$

5 ▶ A garden is $\sqrt{30}$ m long and $\sqrt{8}$ m wide. The garden is covered in grass except for a small rectangular pond which is $\sqrt{2}$ m long and $\sqrt{6}$ m wide.

Express the area of the pond as a percentage of the area of the garden.

6 ▶ Find the value of $2p^2 - 3pq$ when $p = \sqrt{2} + 3$ and $q = \sqrt{2} - 2$

RATIONALISE THE DENOMINATOR OF A SURD

HINT

In the denominator, the multiplication gives the difference of two squares, with the result $a^2 - b^2$, which means the surd disappears.

Rationalising the denominator of a surd means removing a root from the denominator of a fraction. You will usually need to rationalise the denominator when you are asked to *simplify* it.

The rules for rationalising the denominator of a surd are:

- For fractions in the form $\sqrt{\frac{1}{a}}$, multiply the numerator and denominator by $\sqrt{a}$
- For fractions in the form $\frac{1}{a + \sqrt{b}}$, multiply the numerator and denominator by $a - \sqrt{b}$
- For fractions in the form $\frac{1}{a - \sqrt{b}}$, multiply the numerator and denominator by $a + \sqrt{b}$

EXAMPLE 2

Rationalise the denominator of

a $\frac{1}{\sqrt{3}}$ **b** $\frac{1}{3+\sqrt{2}}$ **c** $\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$

a $\frac{1}{\sqrt{3}}$

$= \frac{1 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}}$ Multiply the top and bottom by $\sqrt{3}$

$= \frac{\sqrt{3}}{3}$ $\sqrt{3} \times \sqrt{3} = (\sqrt{3})^2 = 3$

b $\frac{1}{3+\sqrt{2}}$

$= \frac{1 \times (3-\sqrt{2})}{(3+\sqrt{2})(3-\sqrt{2})}$ Multiply the top and bottom by $3-\sqrt{2}$

$= \frac{3-\sqrt{2}}{9-3\sqrt{2}+3\sqrt{2}-2}$ $\sqrt{2} \times \sqrt{2} = 2$

$= \frac{3-\sqrt{2}}{7}$ $9-2=7,\ -3\sqrt{2}+3\sqrt{2}=0$

c $\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$

$= \frac{(\sqrt{5}+\sqrt{2})(\sqrt{5}+\sqrt{2})}{(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})}$ Multiply the top and bottom by $\sqrt{5}+\sqrt{2}$

$= \frac{5+\sqrt{5}\sqrt{2}+\sqrt{2}\sqrt{5}+2}{5-2}$ $\sqrt{5}\sqrt{2}-\sqrt{2}\sqrt{5}=0$ in the denominator

$= \frac{7+2\sqrt{10}}{3}$ $\sqrt{5}\sqrt{2}=\sqrt{10}$

EXERCISE 2

SKILLS

EXECUTIVE FUNCTION

1 ▶ Rationalise

a $\frac{1}{\sqrt{3}}$ **b** $\frac{1}{\sqrt{7}}$ **c** $\frac{2}{\sqrt{3}}$ **d** $\frac{\sqrt{6}}{\sqrt{3}}$

e $\frac{12}{\sqrt{3}}$ **f** $\frac{3\sqrt{5}}{\sqrt{3}}$ **g** $\frac{9\sqrt{12}}{2\sqrt{18}}$ **h** $\frac{1}{2-\sqrt{3}}$

2 ▶ Rationalise

a $\frac{\sqrt{6}}{\sqrt{3}+\sqrt{6}}$ **b** $\frac{2+\sqrt{3}}{2-\sqrt{3}}$ **c** $\frac{\sqrt{2}-\sqrt{3}}{\sqrt{2}+\sqrt{3}}$

d $\frac{4\sqrt{2}-2\sqrt{3}}{\sqrt{2}+\sqrt{3}}$ **e** $\frac{\sqrt{2}+2\sqrt{5}}{\sqrt{5}-\sqrt{2}}$ **f** $\frac{\sqrt{7}+\sqrt{3}}{\sqrt{7}-\sqrt{3}}$

g $\frac{\sqrt{11}+2\sqrt{5}}{\sqrt{11}+3\sqrt{5}}$ **h** $\frac{2\sqrt{5}-3\sqrt{7}}{5\sqrt{6}+4\sqrt{2}}$ **i** $\frac{2+\sqrt{10}}{\sqrt{2}+\sqrt{5}}$

j $\frac{ab}{a\sqrt{b}-b\sqrt{a}}$ **k** $\frac{a-b}{a\sqrt{b}-b\sqrt{a}}$

BE FAMILIAR WITH THE FUNCTIONS a^x AND $\log_a x$ AND RECOGNISE THE SHAPES OF THEIR GRAPHS

You need to be familiar with functions in the form $y = a^x$ where $a > 0$

Look at a table of values for $y = 2^x$

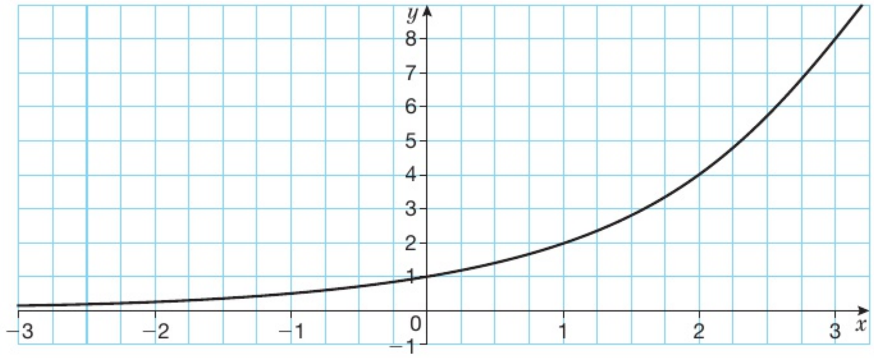
x	–3	–2	–1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Note: $2^0 = 1$

$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

In fact a^0 is always equal to 1 if a is positive and a negative index turns the number into its **reciprocal**

The graph of $y = 2^x$ looks like this:



Note: the x-axis is an **asymptote** to the curve.

Other graphs of the type $y = a^x$ have similar shapes, always passing through (0, 1).

EXAMPLE 3

SKILLS ANALYSIS

a On the same axes, **sketch** the graphs of $y = 3^x$, $y = 2^x$ and $y = 1.5^x$

b On another set of axes, sketch the graphs of $y = \left(\frac{1}{2}\right)^x$ and $y = 2^x$

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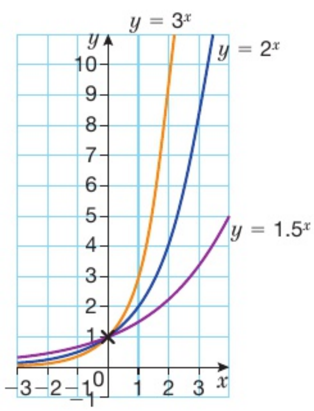
a For all three graphs, $y = 1$ when $x = 0$

$a^0 = 1$

When $x > 0$, $3^x > 2^x > 1.5^x$

When $x < 0$, $3^x < 2^x < 1.5^x$

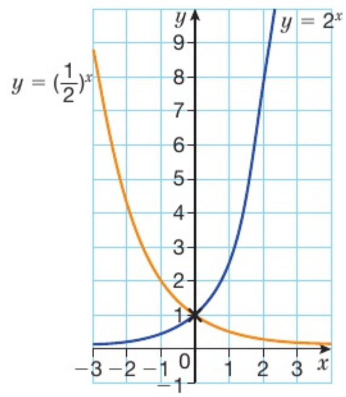
Work out the relative positions of the graphs



b $\frac{1}{2} = 2^{-1}$

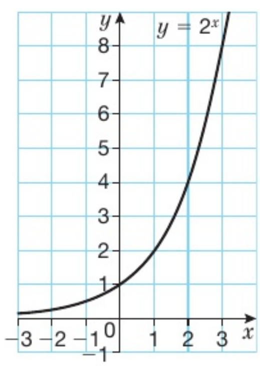
so, $y = \left(\frac{1}{2}\right)^x$ is the same as $y = (2^{-1})^x = 2^{-x}$ $\qquad (a^m)^n = a^{mn}$

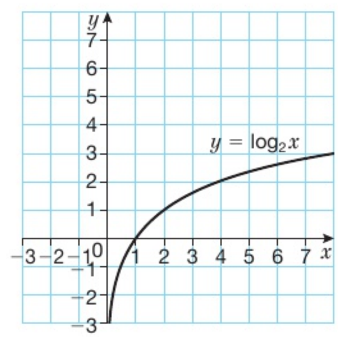
Therefore the graph of $y = \left(\frac{1}{2}\right)^x$ is a reflection in the y-axis of the graph of $y = 2^x$

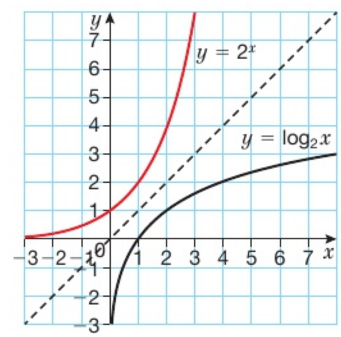


EXAMPLE 4

If you compare the graphs of $y = 2^x$ and $y = \log_2 x$ you see the following relationship:







EXAMPLE 5

On the same set of axes sketch the graphs $y = \log_2 x$ and $y = \log_5 x$

Note:

For both graphs $y = 0$ when $x = 1$, since $\log_a 1 = 0$ for every value of a.

$\log_2 2 = 1$ so $y = \log_2 x$ passes through (2, 1)

and $\log_5 5 = 1$ so $y = \log_5 x$ passes through (5, 1)

EXERCISE 3

SKILLS ANALYSIS REASONING

1 ▶ On the same set of axes sketch the graphs of

a $y = 5^x$ **b** $y = 7^x$ **c** $y = \left(\frac{1}{3}\right)^x$

2 ▶ On the same set of axes sketch the graphs of

a $y = \log_5 x$ **b** $y = \log_7 x$

c Write down the coordinates of the point of intersection of these two graphs.

3 ▶ On the same set of axes sketch the graphs of

a $y = 3^x$ **b** $y = \log_3 x$

4 ▶ On the same set of axes sketch the graphs of

a $y = \log_3 x$ **b** $y = \log_5 x$ **c** $y = \log_{0.5} x$ **d** $y = \log_{0.25} x$

BE FAMILIAR WITH EXPRESSIONS OF THE TYPE e^x AND USE THEM IN GRAPHS

Consider this example: Zainab opens an account with $1.00. The account pays 100% interest per year. If the interest is credited once, at the end of the year, her account will contain $2.00. How much will it contain after a year if the interest is calculated and credited more frequently? Let us investigate this more thoroughly.

HOW OFTEN INTEREST IS CREDITED INTO THE ACCOUNT	VALUE OF ACCOUNT AFTER 1 YEAR ($)
Yearly	$\left(1+\frac{1}{1}\right)^1 = 2$
Semi-annually	$\left(1+\frac{1}{2}\right)^2 = 2.25$
Quarterly	$\left(1+\frac{1}{4}\right)^4 = 2.441406...$
Monthly	$\left(1+\frac{1}{12}\right)^{12} = 2.61303529...$
Weekly	$\left(1+\frac{1}{52}\right)^{52} = 2.69259695...$
Daily	$\left(1+\frac{1}{365}\right)^{365} = 2.71456748...$
Hourly	$\left(1+\frac{1}{8760}\right)^{8760} = 2.71812669...$
Every minute	$\left(1+\frac{1}{525\,600}\right)^{525\,600} = 2.7182154...$
Every second	$\left(1+\frac{1}{31\,536\,000}\right)^{31\,536\,000} = 2.71828247...$

The amount in her account gets bigger and bigger the more often the interest is compounded, but the rate of growth slows. As the number of compounds increases, the calculated value appears to be approaching a fixed value. This value gets closer and closer to a fixed value of 2.71828247254…….This number is called ‘e’.

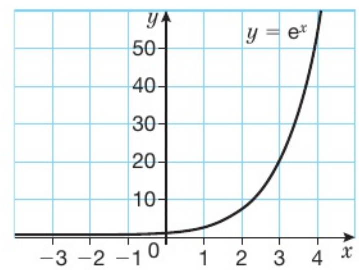
The number e is called a *natural* exponential because it arises *naturally* in mathematics and has numerous real life applications.

EXAMPLE 6

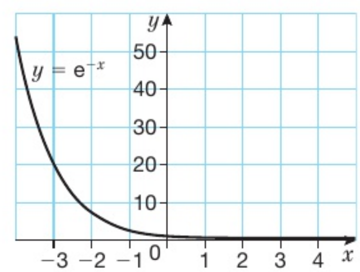
SKILLS
ANALYSIS
INTERPRETATION

Draw the graphs of e^x and e^{-x}

x	−2	−1	0	1	2	3	4
e^x	0.14	0.37	1	2.7	7.4	20	55



x	−4	−3	−2	−1	0	1	2
e^{-x}	55	20	7.4	2.7	1	0.37	0.14



EXAMPLE 7

Draw the graphs of these **exponential functions.**

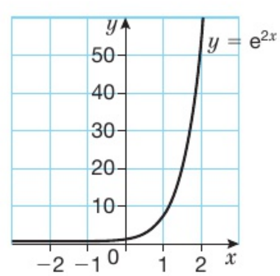
a $y = e^{2x}$

b $y = 10e^{-x}$

c $y = 3 + 4e^{\frac{1}{2}x}$

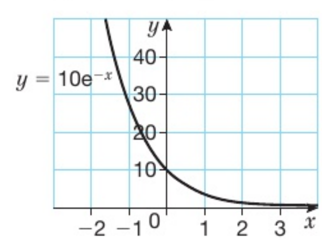
a

x	−2	−1	0	1	2
e^{2x}	0.02	0.1	1	7.4	55



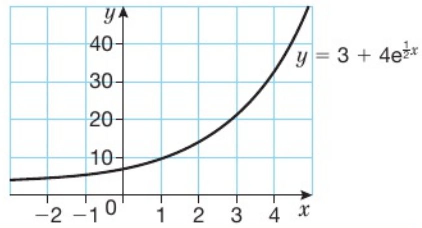
b

x	−2	−1	0	1	2
$10e^{-x}$	73	27	10	3.7	1.4



c

x	-2	-1	0	1	2
$3 + 4e^{\frac{1}{2}x}$	4.5	5.4	7	8.9	13.4



On pages 7–9 you saw the connection between $y = \log_a x$ and and $y = a^x$. The function $y = \log_e x$ is particularly important in mathematics and so it has a special notation:

$$\log_e x \equiv \ln x$$

Your calculator should have a special button for evaluating $\ln x$.

EXAMPLE 8

Solve these equations.

a $e^x = 3$ **b** $\ln x = 4$

a When $e^x = 3$
$x = \ln 3$

b When $\ln x = 4$
$x = e^4$

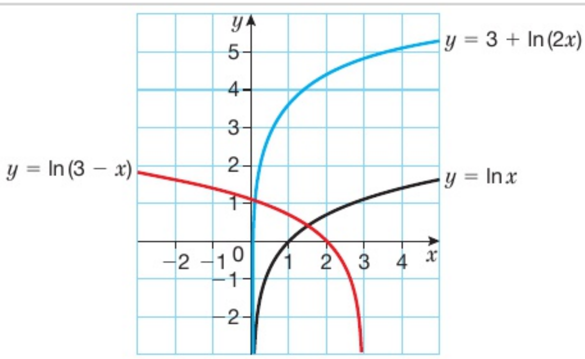
As you can see, the inverse of e^x is $\ln x$ (and vice versa)

EXAMPLE 9

Sketch these graphs on the same set of axes.

a $y = \ln x$ **b** $y = \ln(3 - x)$ **c** $y = 3 + \ln(2x)$

a–c



EXERCISE 4

SKILLS ANALYSIS REASONING

1 ▶ Sketch these graphs.

a $y = e^x + 1$ **b** $y = 4e^{-2x}$ **c** $y = 2e^x - 3$

d $y = 6 + 10^{\frac{1}{2}x}$ **e** $y = 100e^{-x} + 10$

2 ▶ Sketch these graphs, stating any asymptotes and intersections with the axes.

a $y = \ln(x + 1)$ **b** $y = 2\ln x$ **c** $y = \ln(2x)$

d $y = \ln(4 - x)$ **e** $y = 4 + \ln(x + 2)$

BE ABLE TO USE GRAPHS OF FUNCTIONS TO SOLVE EQUATIONS

EXAMPLE 10

a Complete the table of values for: $y = e^{\frac{1}{2}x} - 2$
Giving your answers to two decimal places where appropriate.

b Draw the graph of $y = e^{\frac{1}{2}x} - 2$ for $0 \leqslant x \leqslant 5$

c Use your graph to estimate, to 2 significant figures, the solution of the equation $e^{\frac{1}{2}x} = 8$
Show your method clearly.

d By drawing a suitable line on your graph, estimate to 2 significant figures the solution to the equation $x = 2\ln(7 - 2x)$

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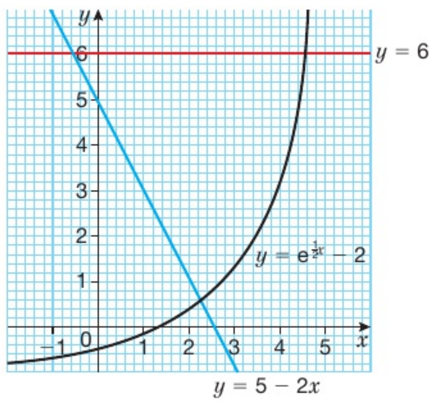
a

x	−1	0	1	2	3	4	5
y	−1.39	−1	−0.35	0.72	2.48	5.39	10.18

b



HINT
Make the LHS = $e^{\frac{1}{2}x} - 2$ i.e. the equation of the graph. To do this, you need to subtract 2 from 8 and draw the line $y = 6$ (as shown in the diagram).

c

$$e^{\frac{1}{2}x} = 8$$
$$e^{\frac{1}{2}x} - 2 = 8 - 2$$
$$e^{\frac{1}{2}x} = 6$$

So the solution is the intersection of the curve $y = e^{\frac{1}{2}x}$ and $y = 6$
$x \approx 4.15$ (In the exam you will be allowed a range of values.)

HINT
Make the **LHS** equal to the given equation i.e $e^{\frac{1}{2}x} - 2$. Draw the line $y = 5 - 2x$ (as shown in the diagram) on your graph and find points of intersection.

d

$$x = 2\ln(7 - 2x)$$
$$\frac{x}{2} = \ln(7 - 2x)$$
$$e^{\frac{1}{2}x} = 7 - 2x$$
$$e^{\frac{1}{2}x} - 2 = 7 - 2x - 2$$
$$e^{\frac{1}{2}x} = 5 - 2x$$

So, the solution is the intersection of the curve and $y = e^{\frac{1}{2}x}$ and the line $y = 5 - 2x$, $x \approx 2.1$

EXAMPLE 11

a Complete the table below of values of $y = 2 + \ln x$, giving your values of y to decimal places.

x	0.1	0.5	1	1.5	2	3	4
y	−0.3	1.31		2.41	2.69	3.10	